Efficient computation of distances between FEM solutions defined on different 3D meshes

CHANGE workshop, Leysin February 2018

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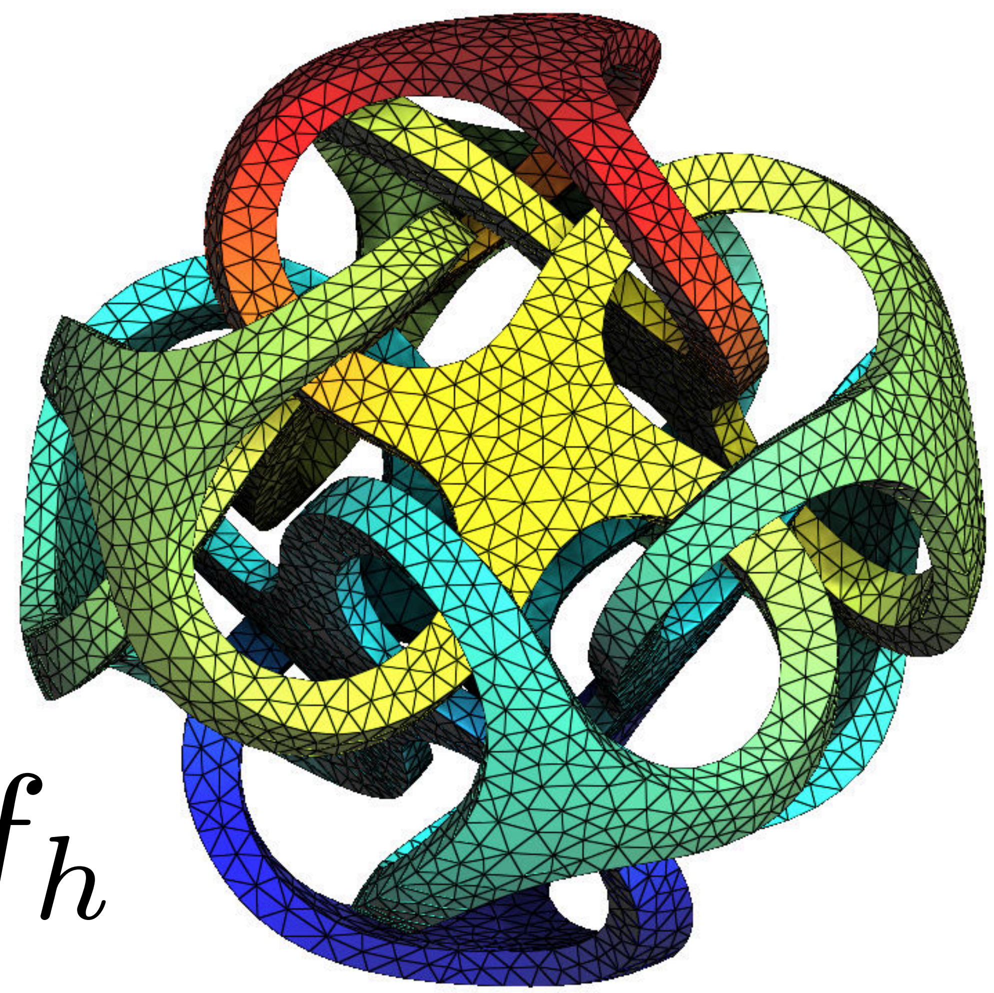
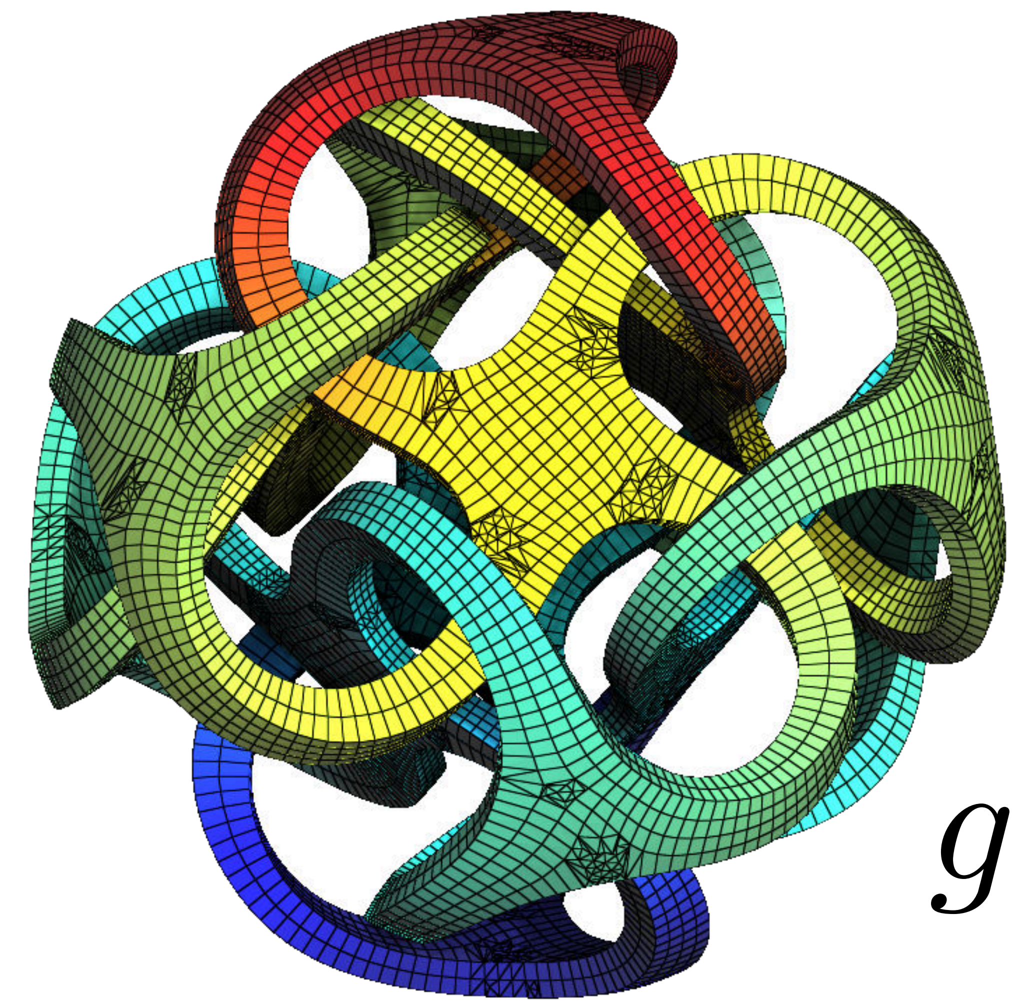
ALICE team, Inria Grand-Est

Distance between FEM solutions



f_h



g_h

Subject of the talk:

How to compute $||f_h - g_h||_{L^2} = \sqrt{\int_\Omega (f_h - g_h)^2}$?

Motivations

How to evaluate FEM accuracy and performance ?
(influence of meshes, refinement, order, etc)

1. If analytical solution is known:
   - compute error (L^2, H^1, ..) with quadratures

   - problems can be built with *Method of Manufactured Solutions*:

     a) choose analytical solution $u(x, y) = \sin(2\pi x)\sin(2\pi y)$

     b) inject in problem (domain + PDE + BCs) $-\Delta u = f$

     c) derive formula for source term and BCs $f(x, y) = 4\pi^2 \sin(2\pi x)\sin(2\pi y)$

   [Salari00, Roache02]

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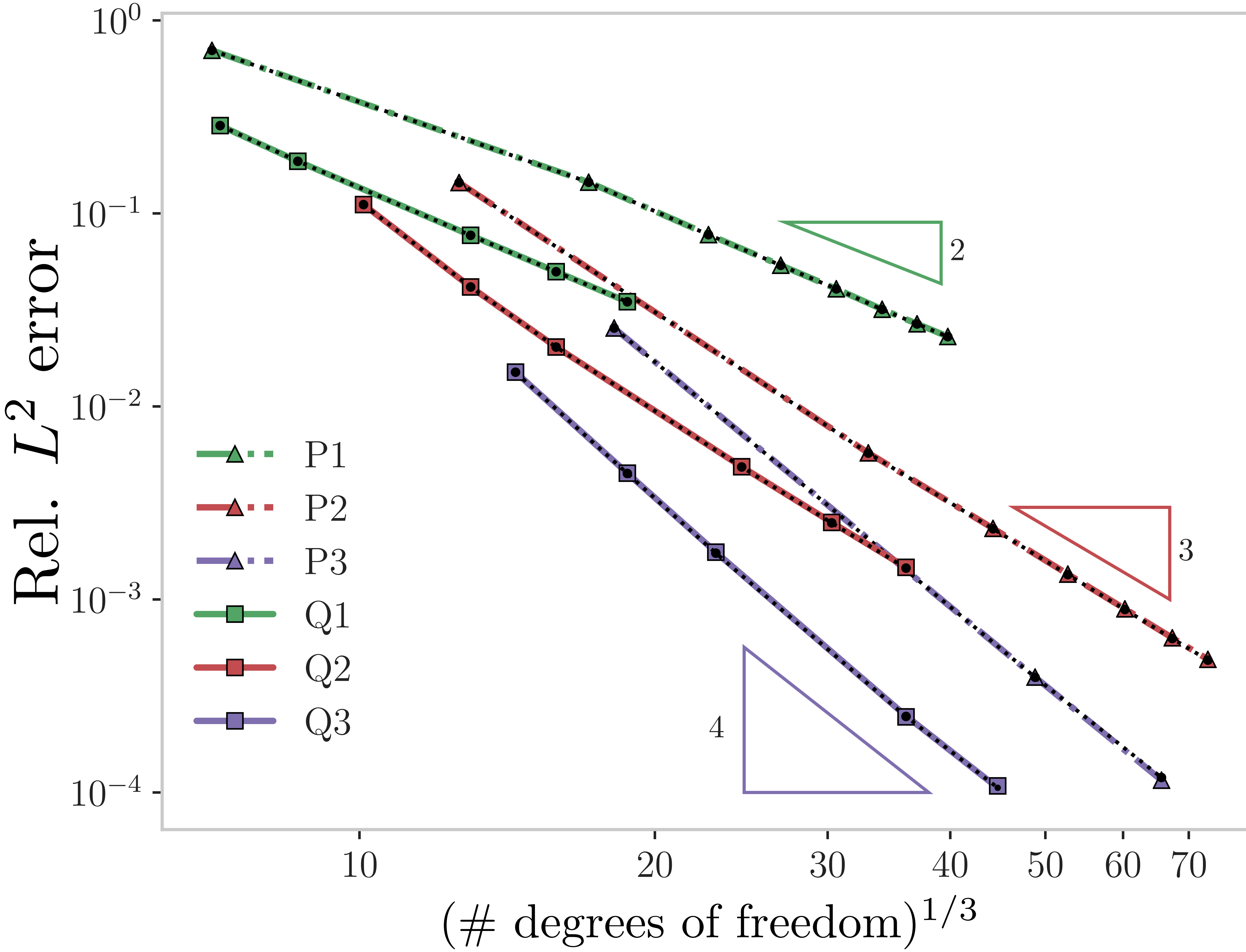
Motivations: evaluation of FEM accuracy

1. If analytical solution, convergence analysis is easy:



Motivations: evaluation of FEM accuracy

1. If analytical solution, convergence analysis is easy..
   but results are not representative of real-life performance or accuracy:

   - very simple domains (unit cube usually)

   - analytical RHS everywhere, no *propagation* from boundaries (contrary to real-life problems where RHS is null/constant)

   - measuring source term approximation (or coefficients, BCs, etc)
     e.g. $-\Delta u = f$, **f** not in the approximation space

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Motivations: evaluation of FEM accuracy

2. If no analytical solution, use a reference solution

For specific applications, compare relevant quantities:
- maximum stress (mechanics), drag coefficient (aerodynamic), eigenvalues, etc

For general purpose, how to compute (L^2,H^1,...) errors ?
- approximate error with distance to reference solution

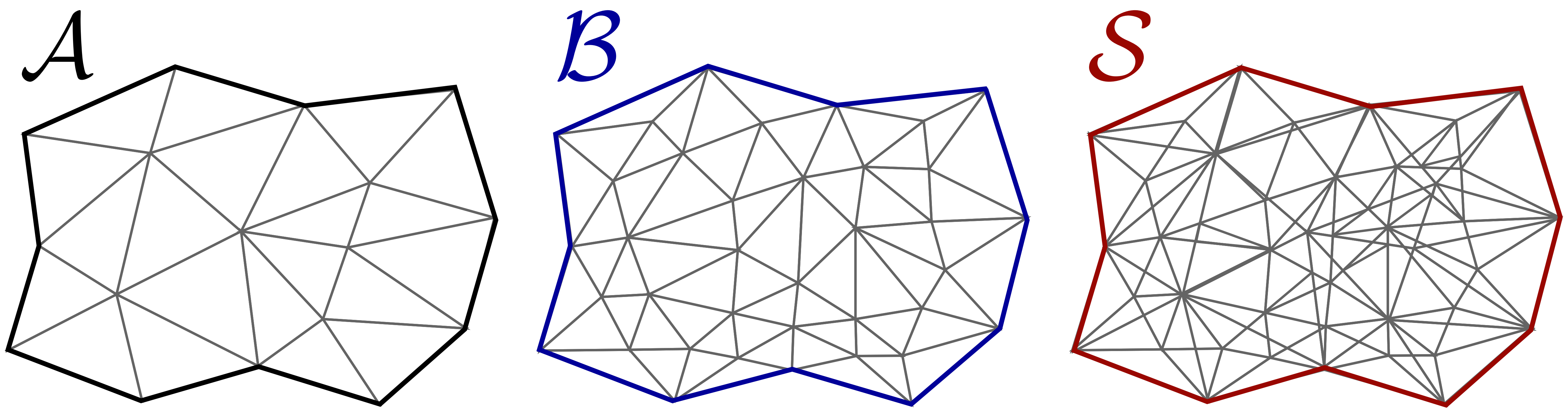
New question: how to compute distances ?

Distance computation

Integral replaced by weighted sum:

$$(||f_h - g_h||_{L^2})^2 = \int_\Omega (f_h - g_h)^2 \approx \sum_{i=1}^{N} w_i (f_h(x_i) - g_h(x_i))^2$$

Proper way: it's FEM, so let's use quadratures
(requires mesh and local interpolation)



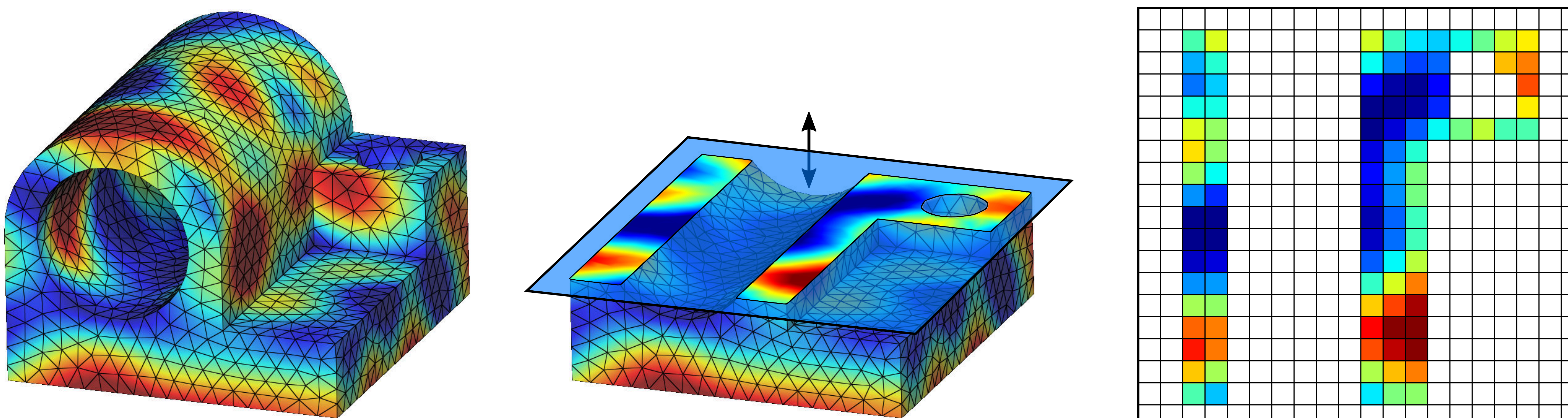
Bibliography:
- supermesh
- rendezvous mesh
- intersection mesh

3D with curved cells (hexahedra, quadratic tets, etc) ?

Distance computation with regular sampling

Computer graphics approach: 

- sampling of both fields on a (large) regular voxel grid (typical size: 1000^3)
- slice by slice (low memory consumption)



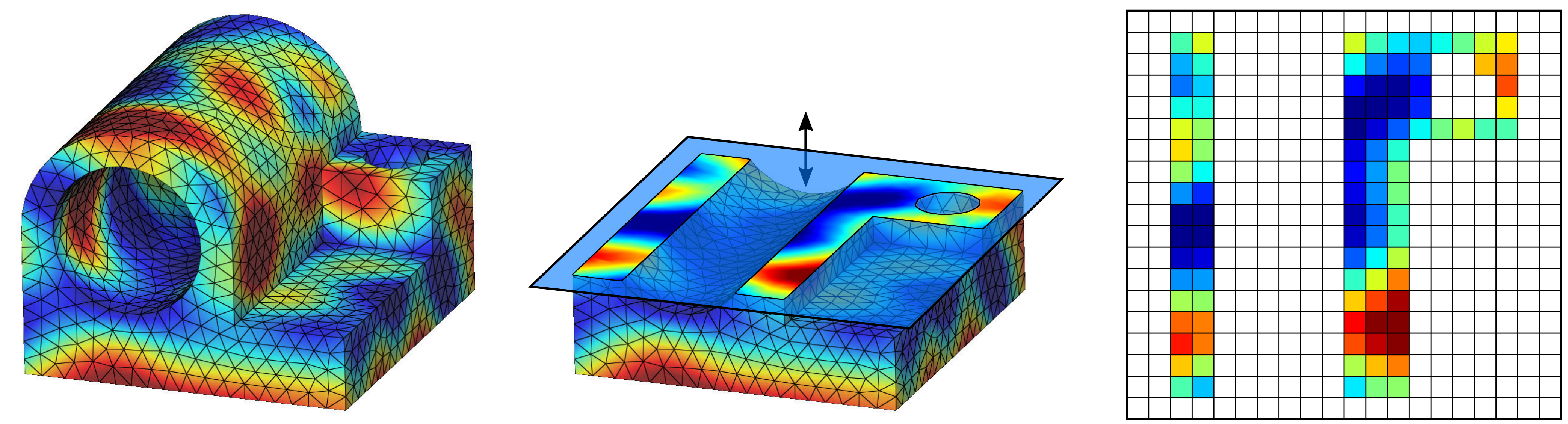
(slice of a FEM field)

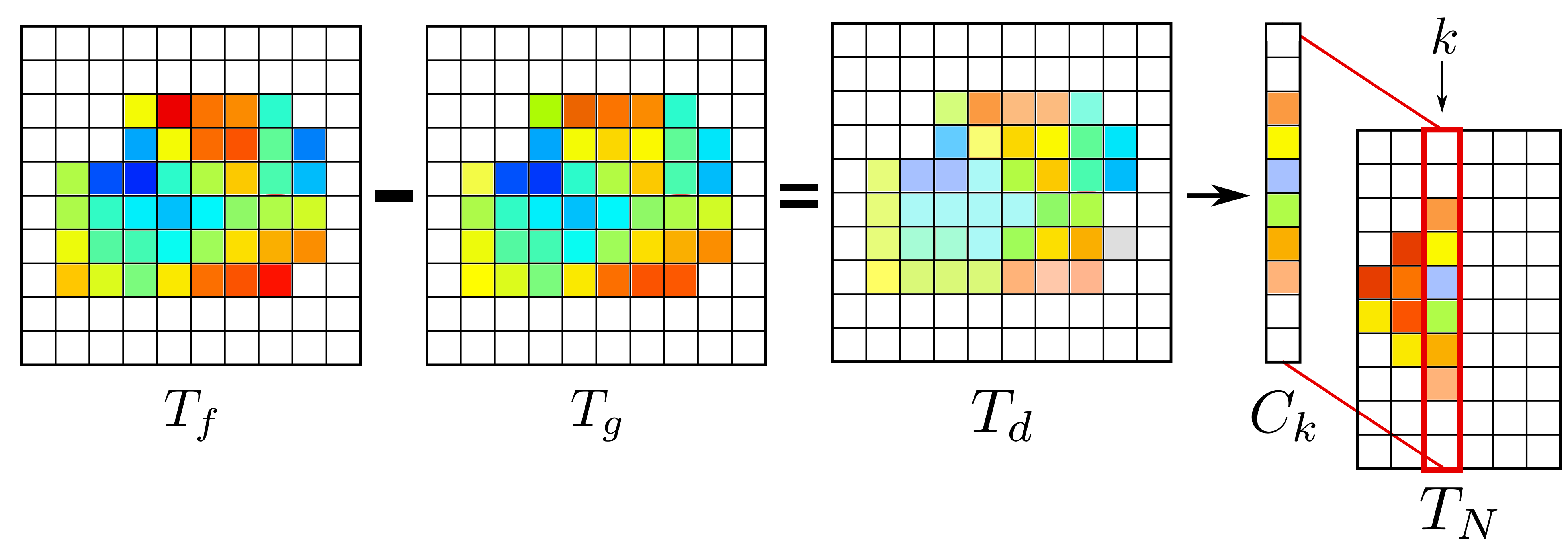
Distance computation: global algorithm

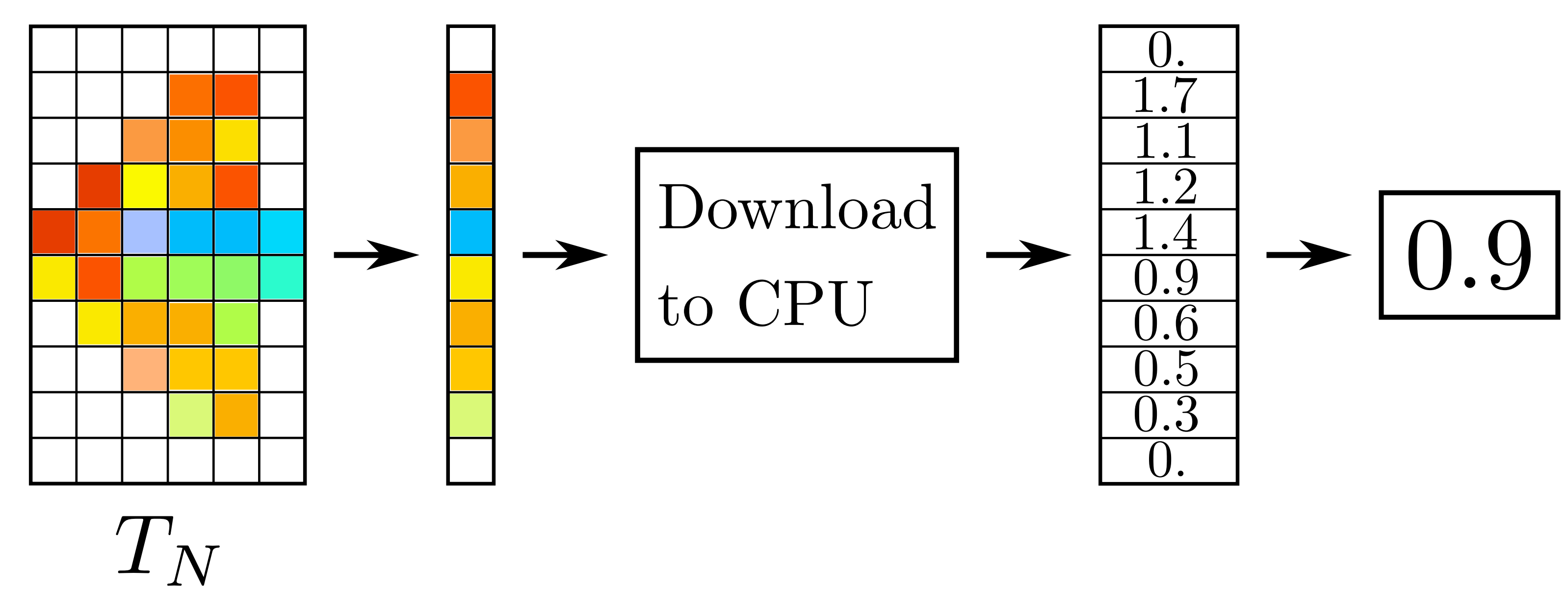
1. Initialization
   (upload data to GPU)

2. For each slice:
   - render field A
   - render field B
   - compute difference
   - store contribution to distance

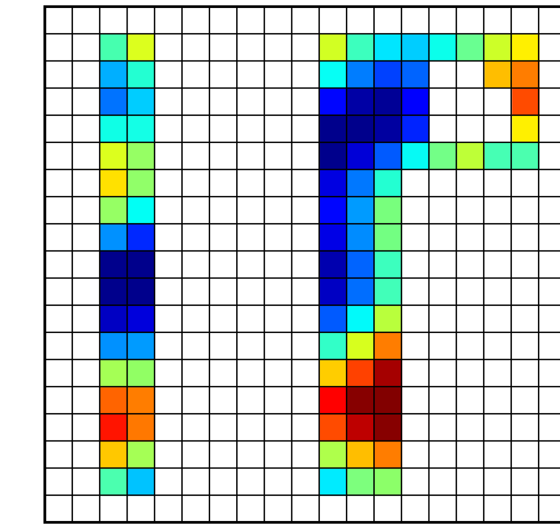
3. Combine contributions to get global distance



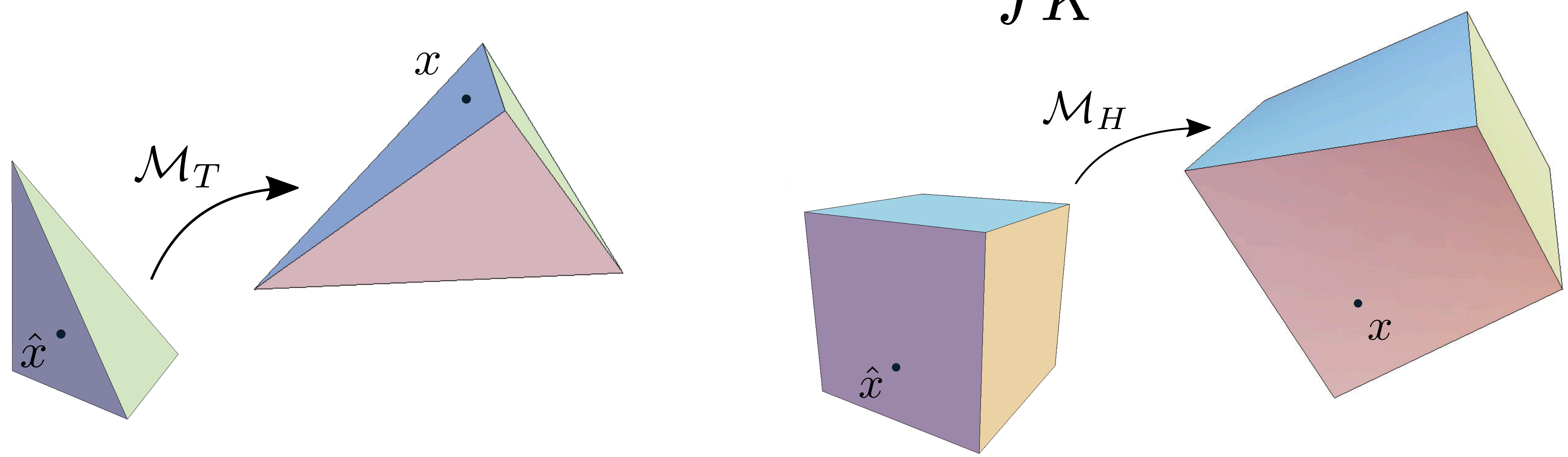




Computing the field values

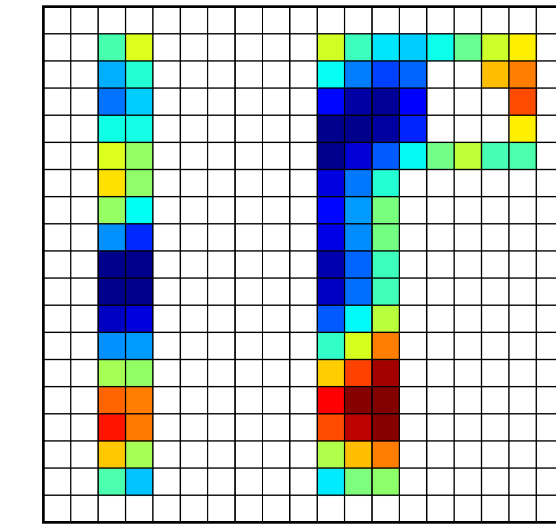


FEM interpolation is defined per cell:
- mapping from reference element to world space $\mathcal{M}_K$
- interpolation defined in reference element $\hat{f}_K$



At each pixel: $$f_h(x) = \hat{f}_K \circ \underbrace{\mathcal{M}_K^{-1}(x)}_{\text{reference coordinates } \hat{x}} \text{ for } x \in K$$

Computing the field values

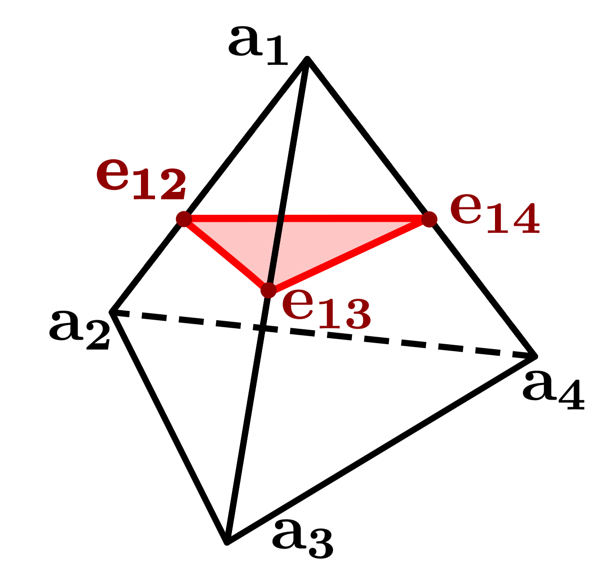


$$f_h(x) = \hat{f}_K \circ \mathcal{M}_K^{-1}(x) \text{ for } x \in K$$

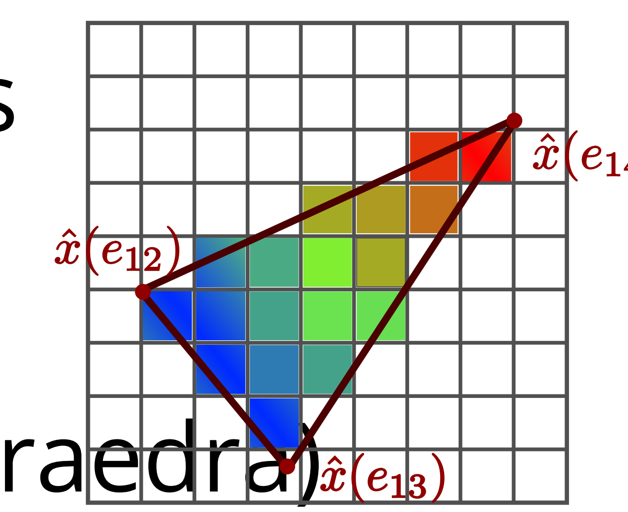
shape functions evaluated
exactly in the fragment shader

(pixel exact rendering
e.g. [BH04, NHK11])

- reference coordinates at triangle vertices with marching tetrahedra [AK91]



- rasterization for linear interpolation at pixel centers



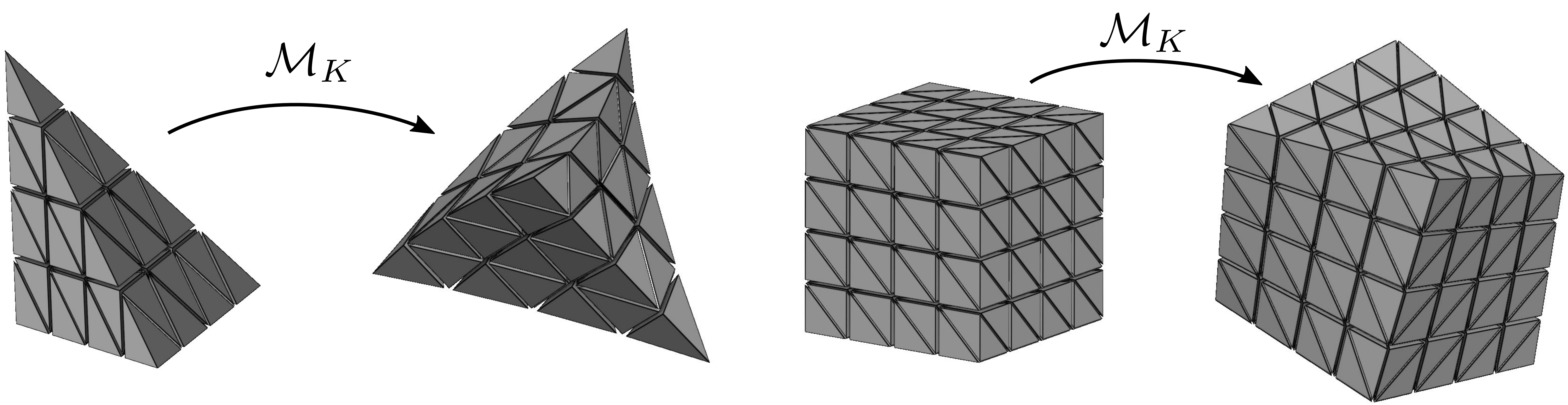
no mapping inversion
exact for linear mappings (tetraedra)

Curved (non-affine) elements

$$f_h(x) = \hat{f}_K \circ \mathcal{M}_K^{-1}(x) \text{ for } x \in K$$

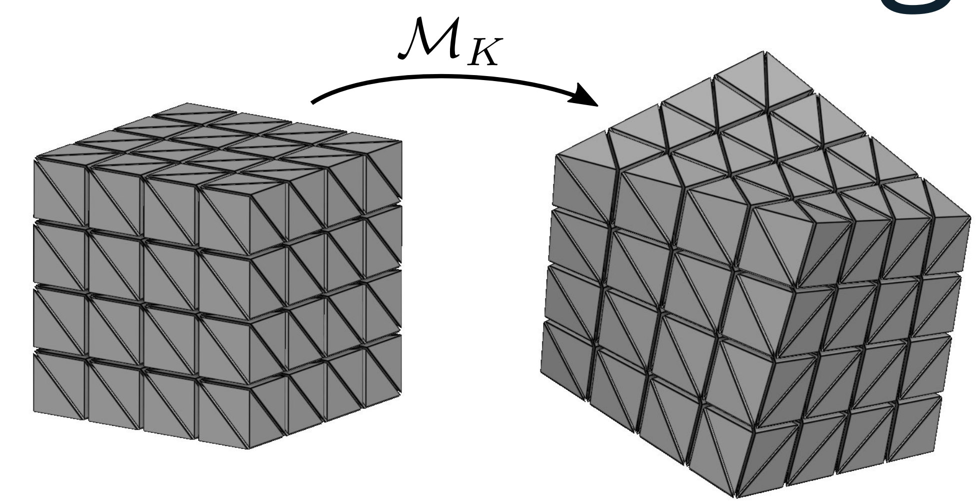
$\mathcal{M}_K$ is approximated by a piecewise-linear mapping

i.e. curved geometry approximated by subdivision
reference coordinates, used in evaluation, are no longer exact

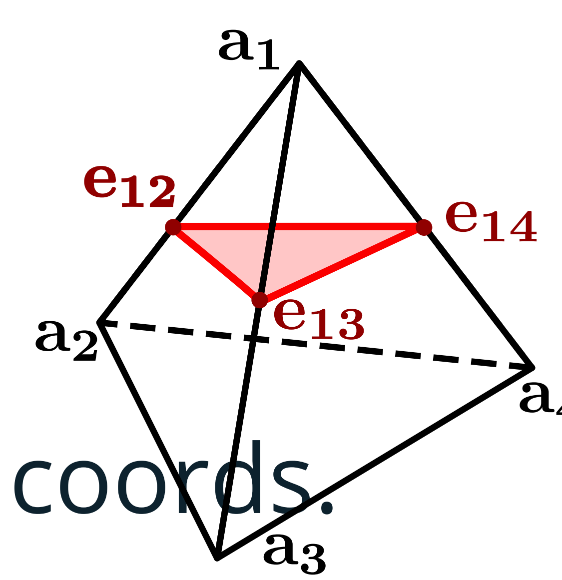


(done on the GPU using the *instance rendering* feature of OpenGL)

Summary of the OpenGL rendering

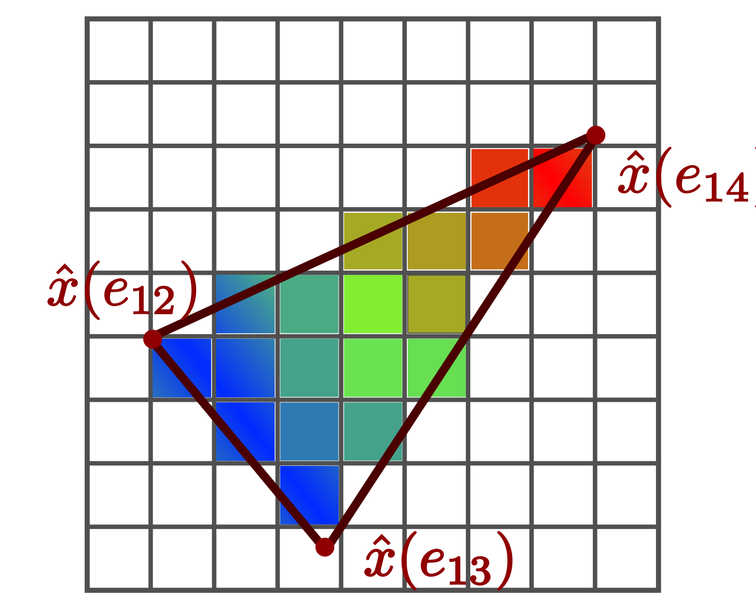


- Vertex shader: mapping
  decomposition if curved
  (mesh coefficients via VertexAttributes)

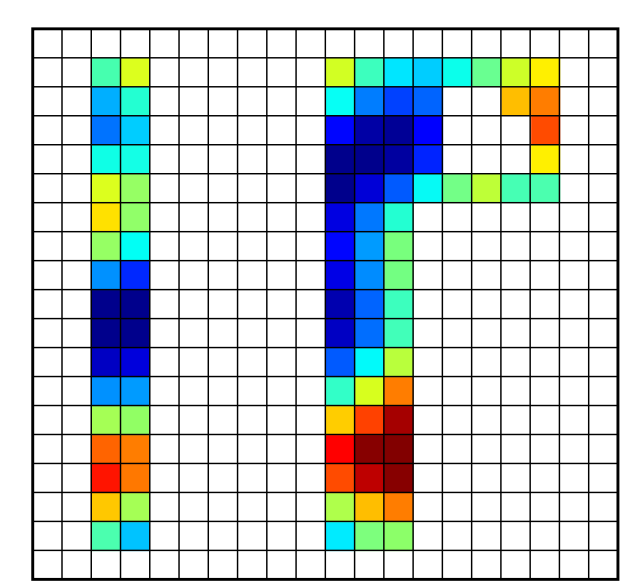


- Geometry shader: marching tetrahedra

- Rasterization (OpenGL): linear interpolation of ref. coords.

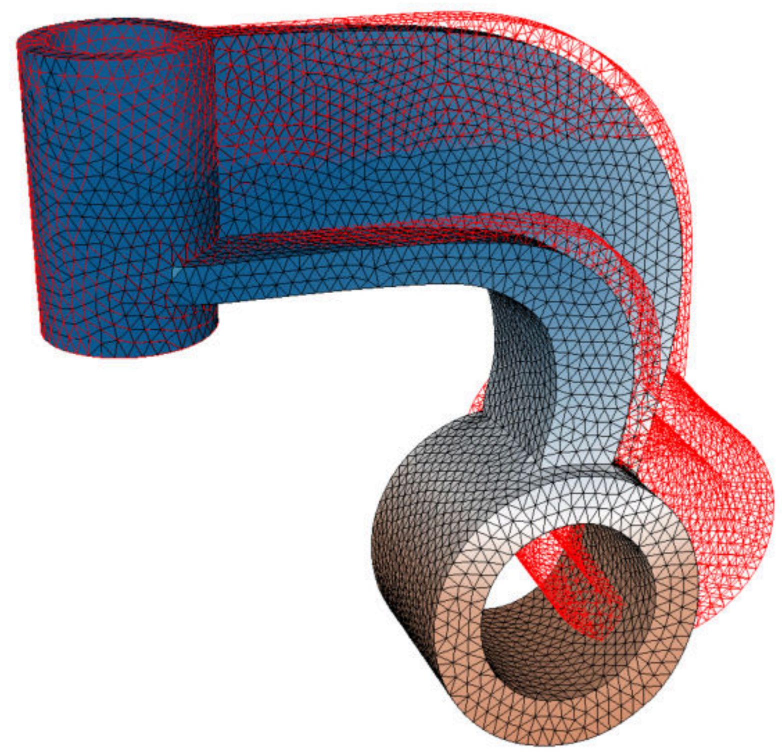


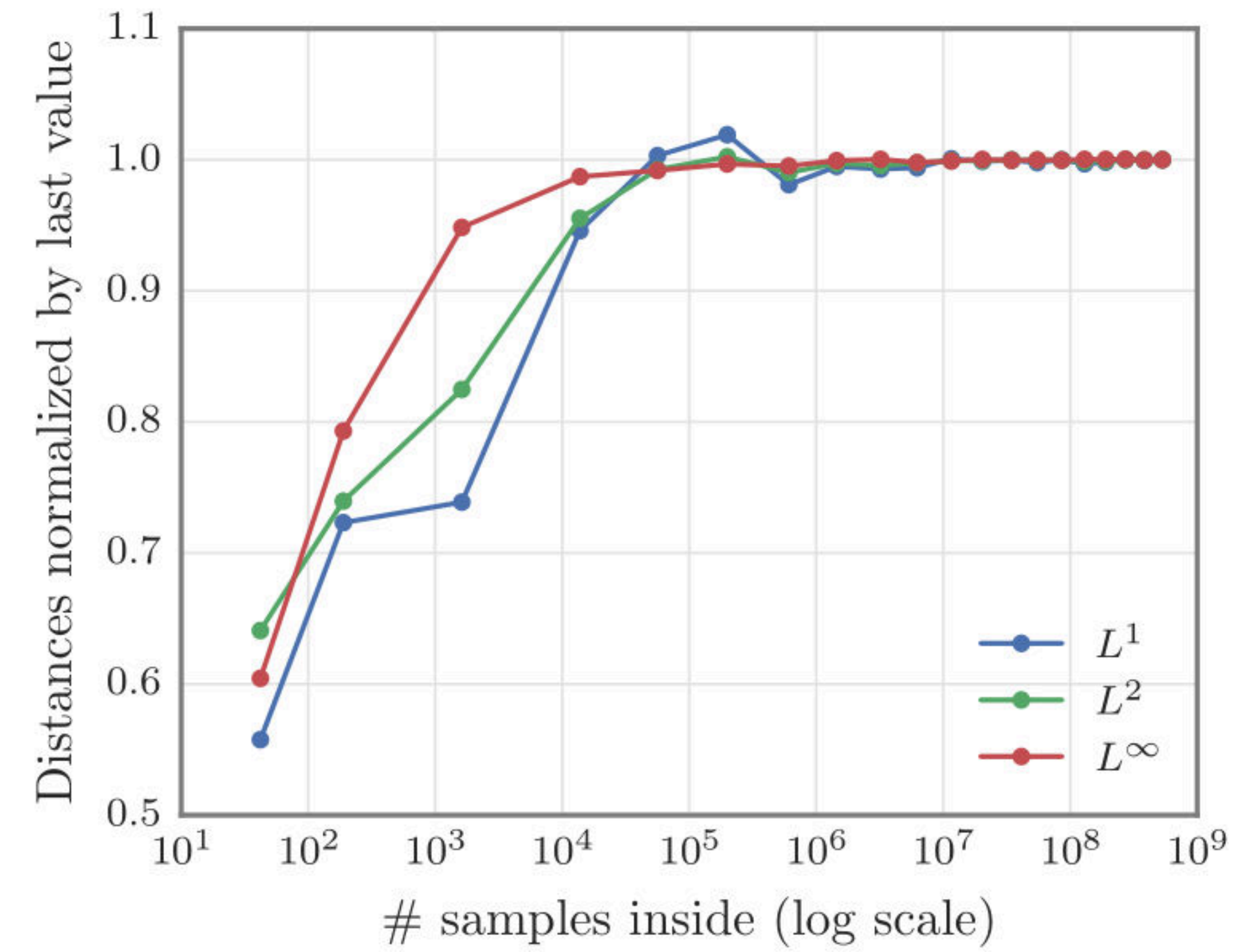
- Fragment shader: shape functions evaluation
  (field coefficients via SSBO)



Parameter sensitivity

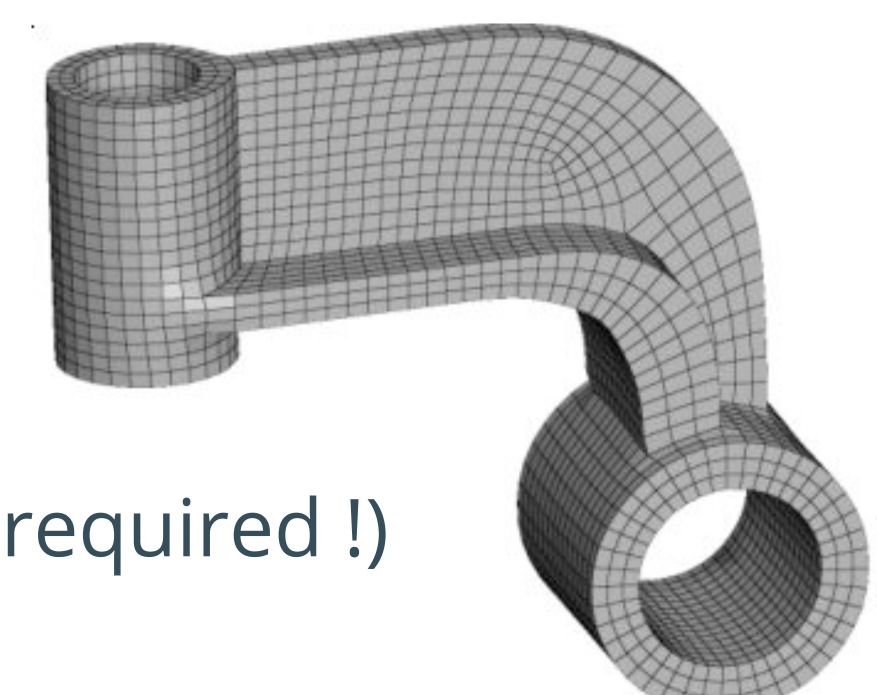
- # samples (voxel grid resolution):



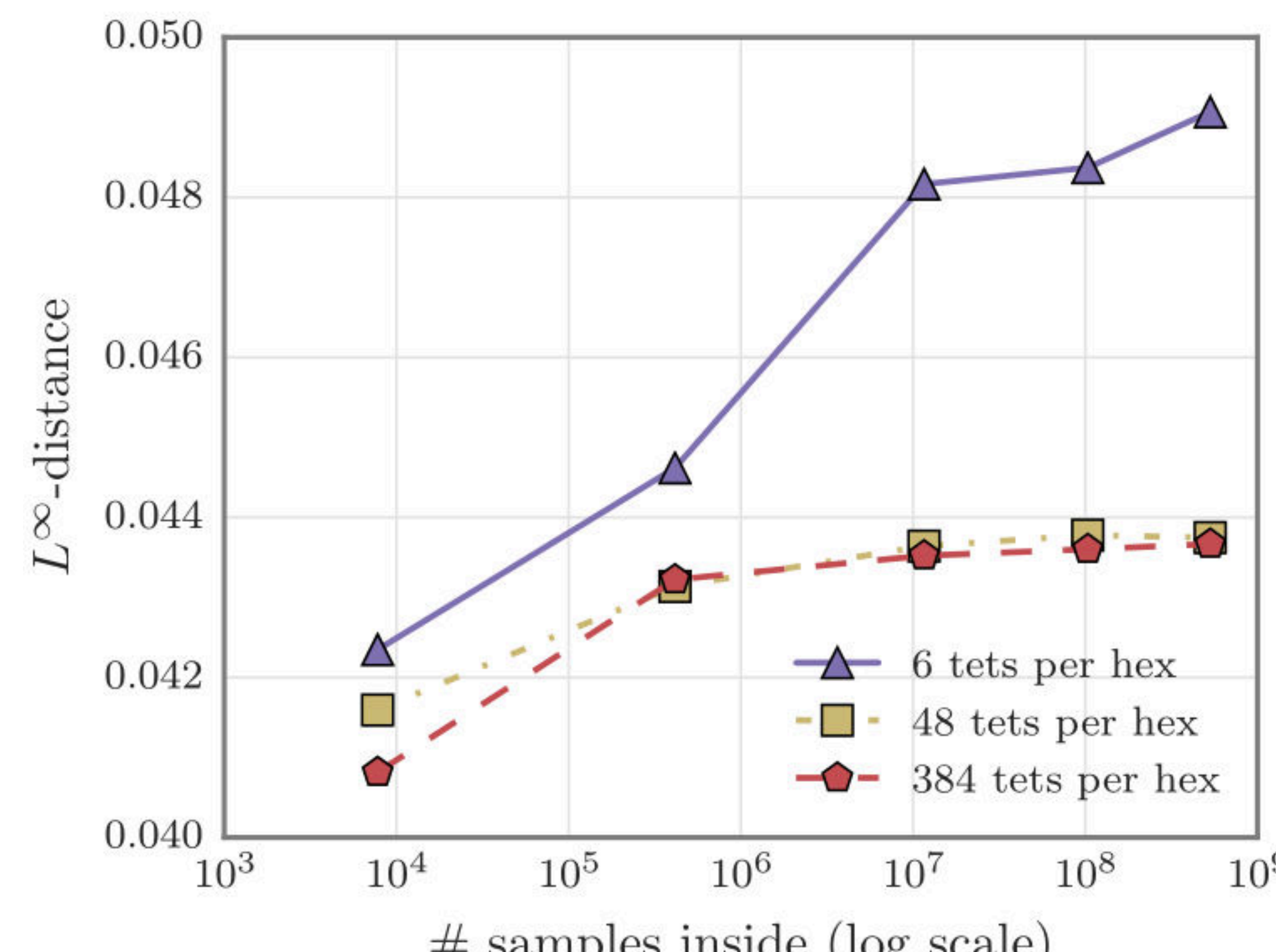


(10-100 millions samples inside at worst)

- approximation of curved cells:



(subdivision required !)



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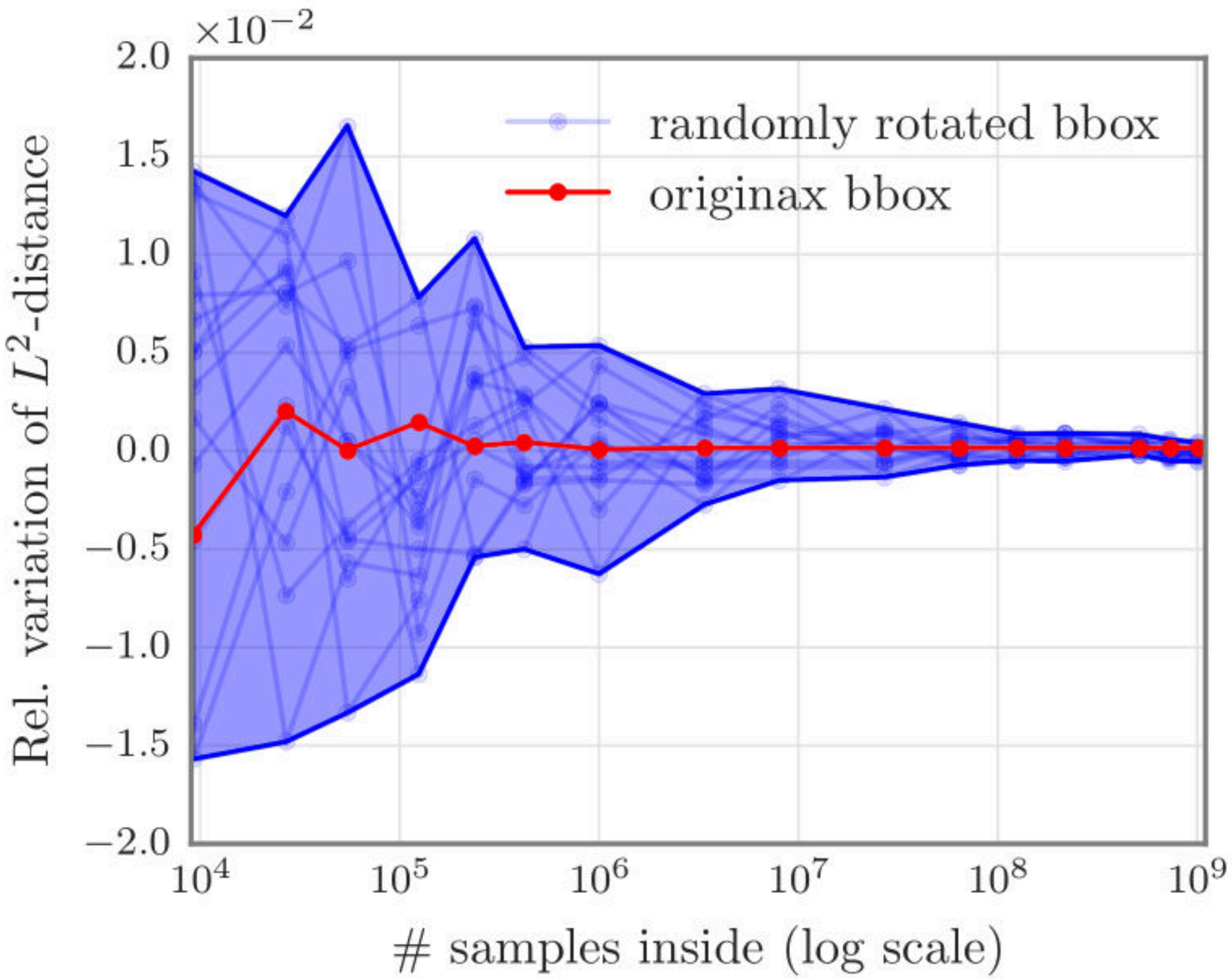
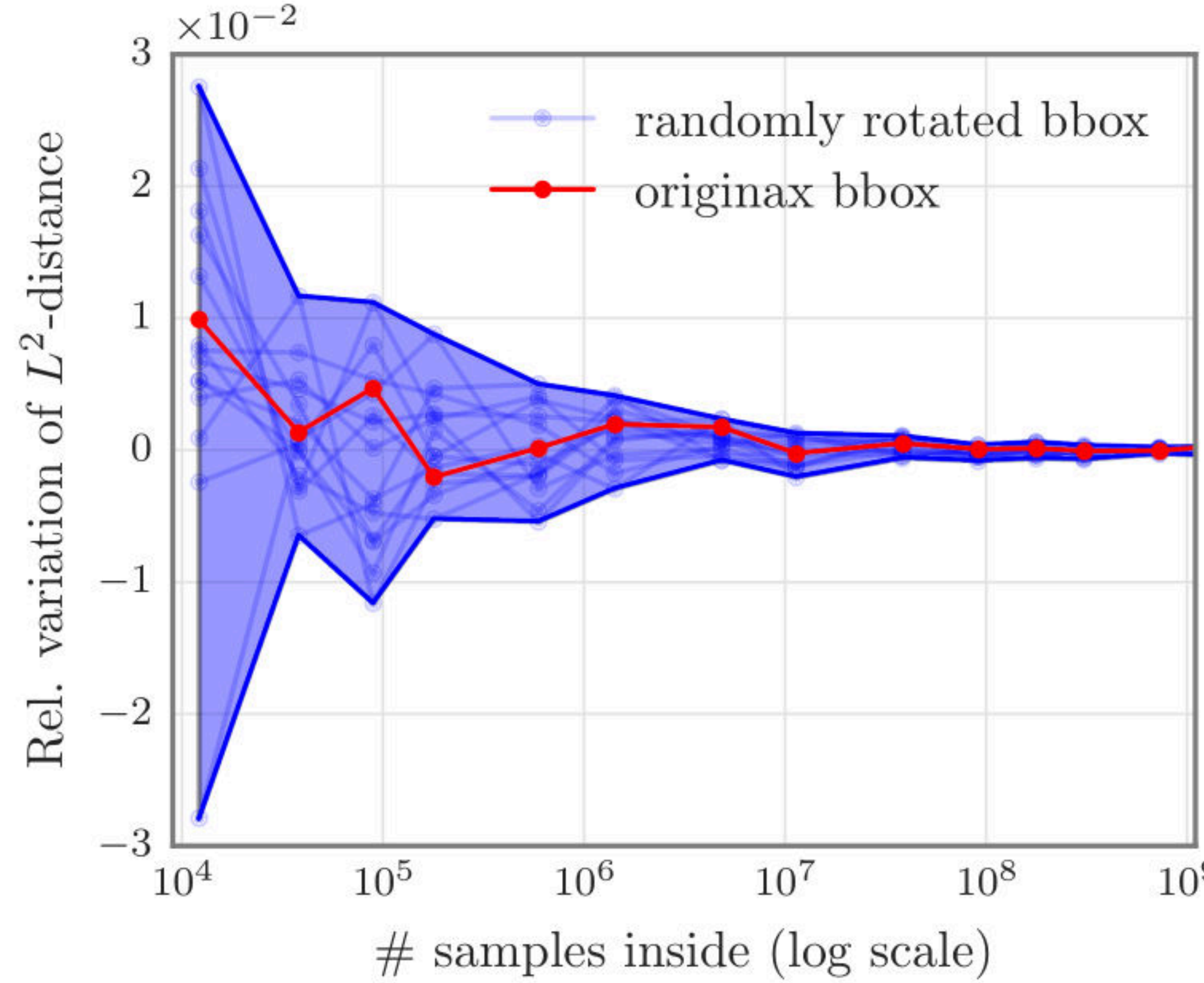
Parameter sensitivity

- voxel grid orientation (two examples):





(ok after a few millions of samples, base orientation is usally good)

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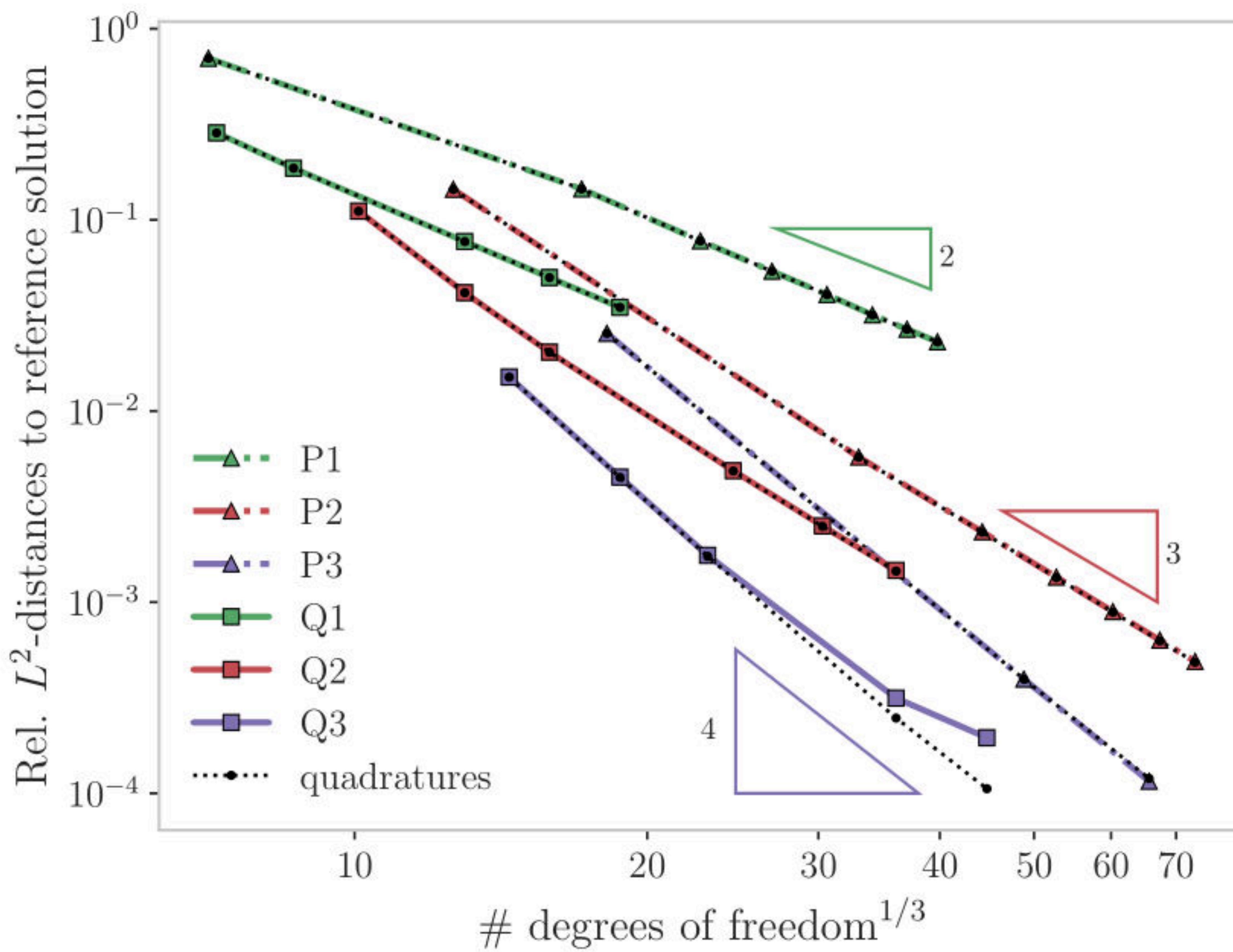
Validation

- linear elasticity problem with analytical solution (built with MMS)
- exact errors computed with high-order quadratures (dotted lines)
- distances to reference solution (fine mesh with P4) for various meshes and orders



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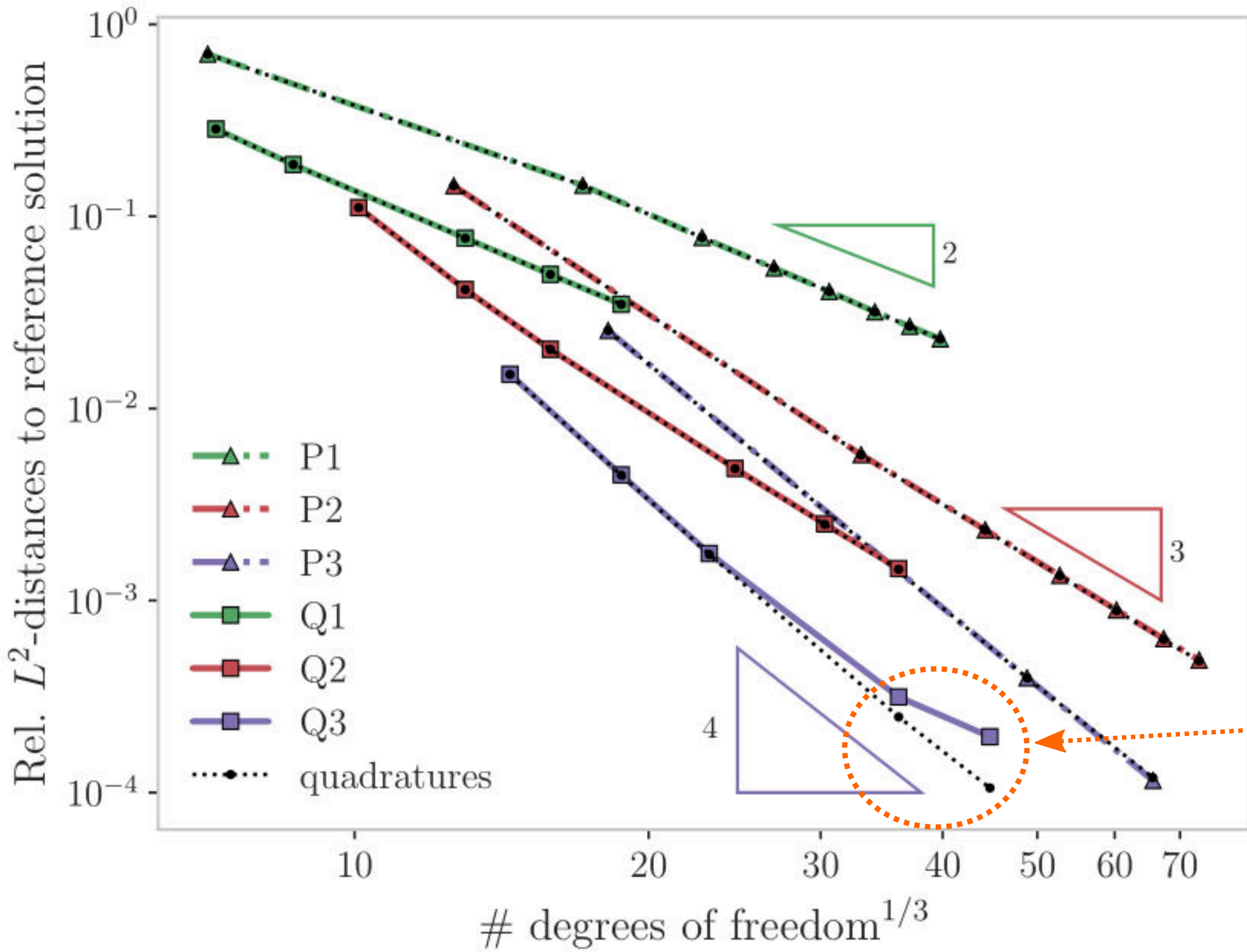
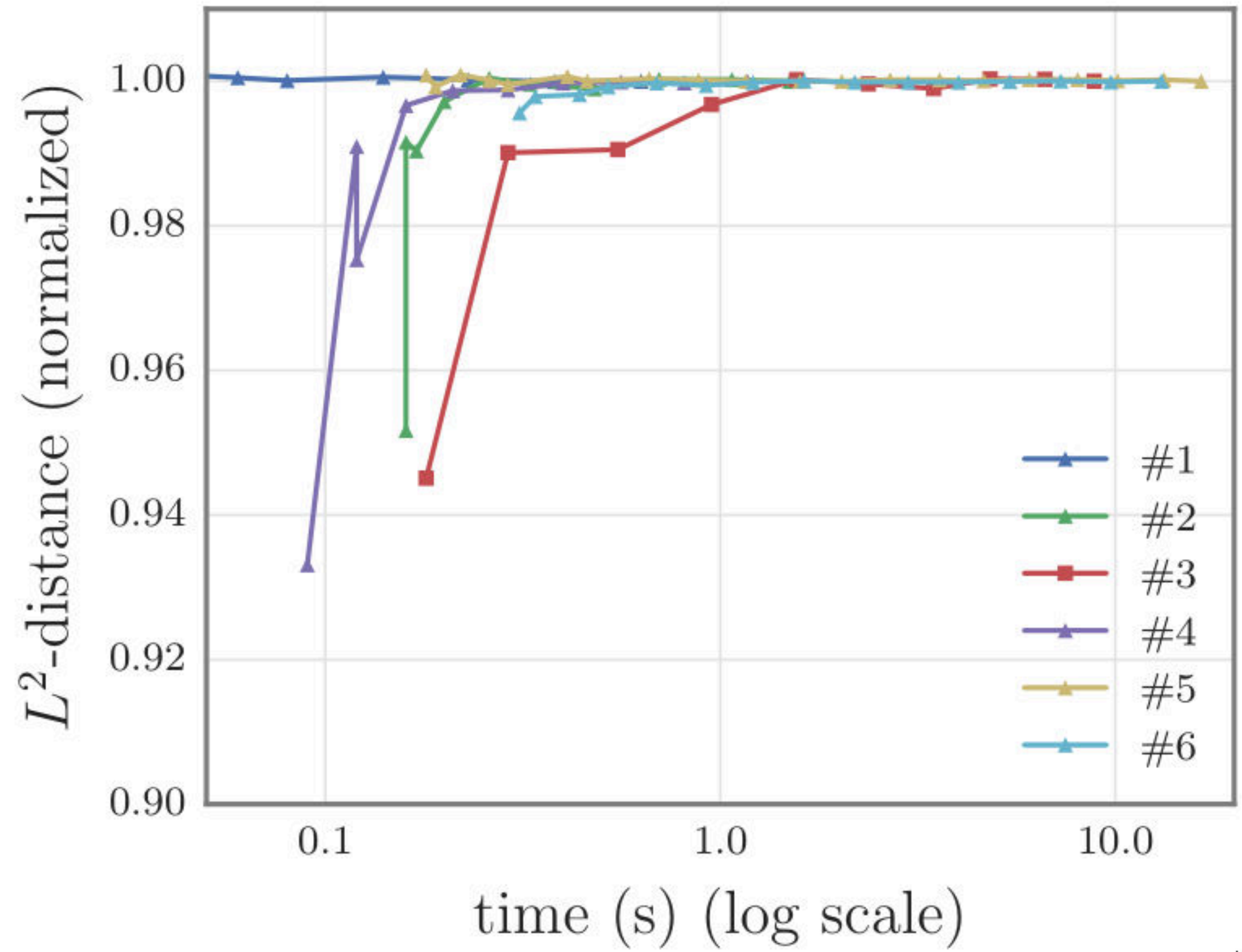


insufficient hex decomposition

Performance

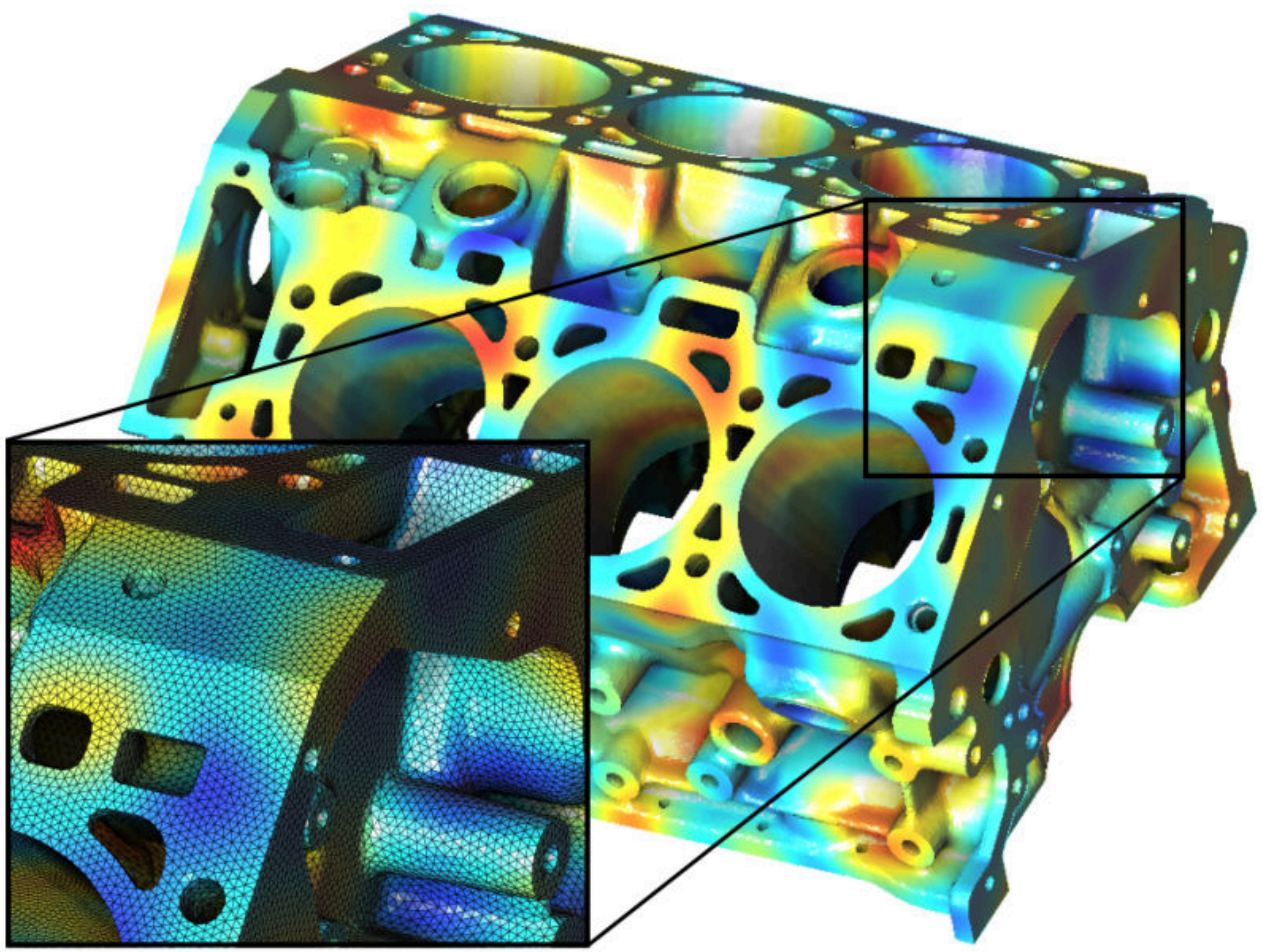
model	dim	# cells	basis
#1 : joint (fig. 5.3)	1	10 k	P_2 (tet)
		65 k	P_1 (tet)
#2 : hanger (fig. 5.10)	3	1 739 k	P_1 (tet)
		343 k	P_2 (tet)
#3 : hanger (fig. 5.10)	3	36 k	Q_2 (hex)
		290 k	Q_1 (hex)
#4 : carter (fig. 5.12)	1	1 210 k	P_1 (tet)
		373 k	P_2 (tet)
#5 : 747 (fig. 5.14)	1	1 385 k	P_1 (tet)
		577 k	P_2 (tet)
#6 : 40heads (fig. 5.14)	1	2 905 k	P_1 (tet)
		2 350 k	P_2 (tet)

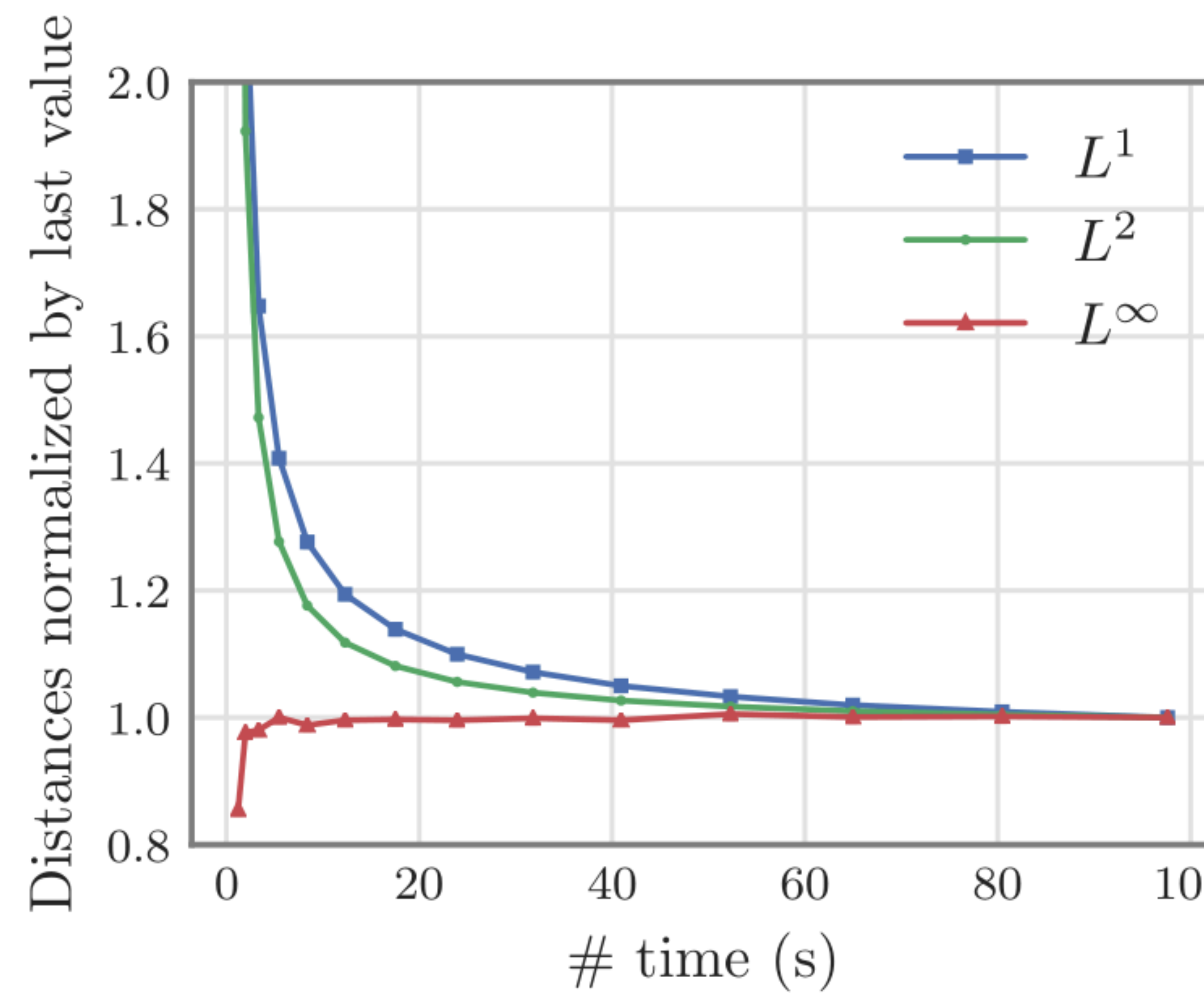


Less than one second for standard meshes (~ 1 million cells)
(timings obtained with Nvidia 1080 GTX)

Performance on large fields

Field A: 2905k tets P5, 64M dofs
Field B: 2305k tets P7, 146M dofs
sparse model structure: ~20% of bbox

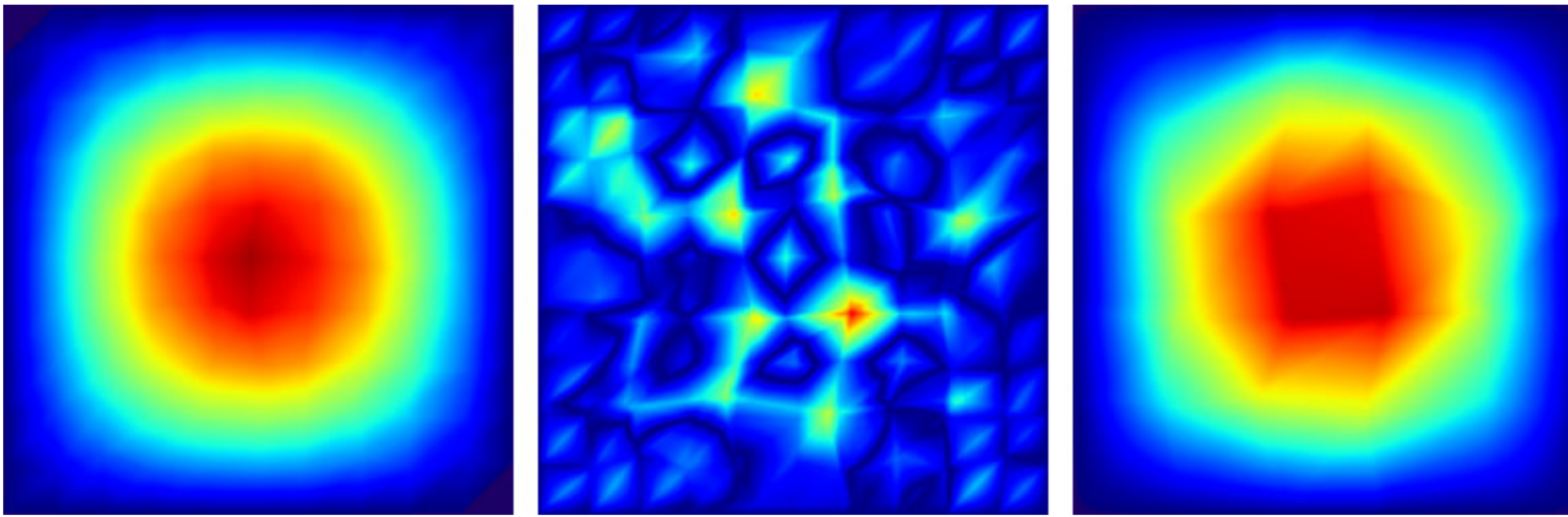




convergence: ~ 1 minute

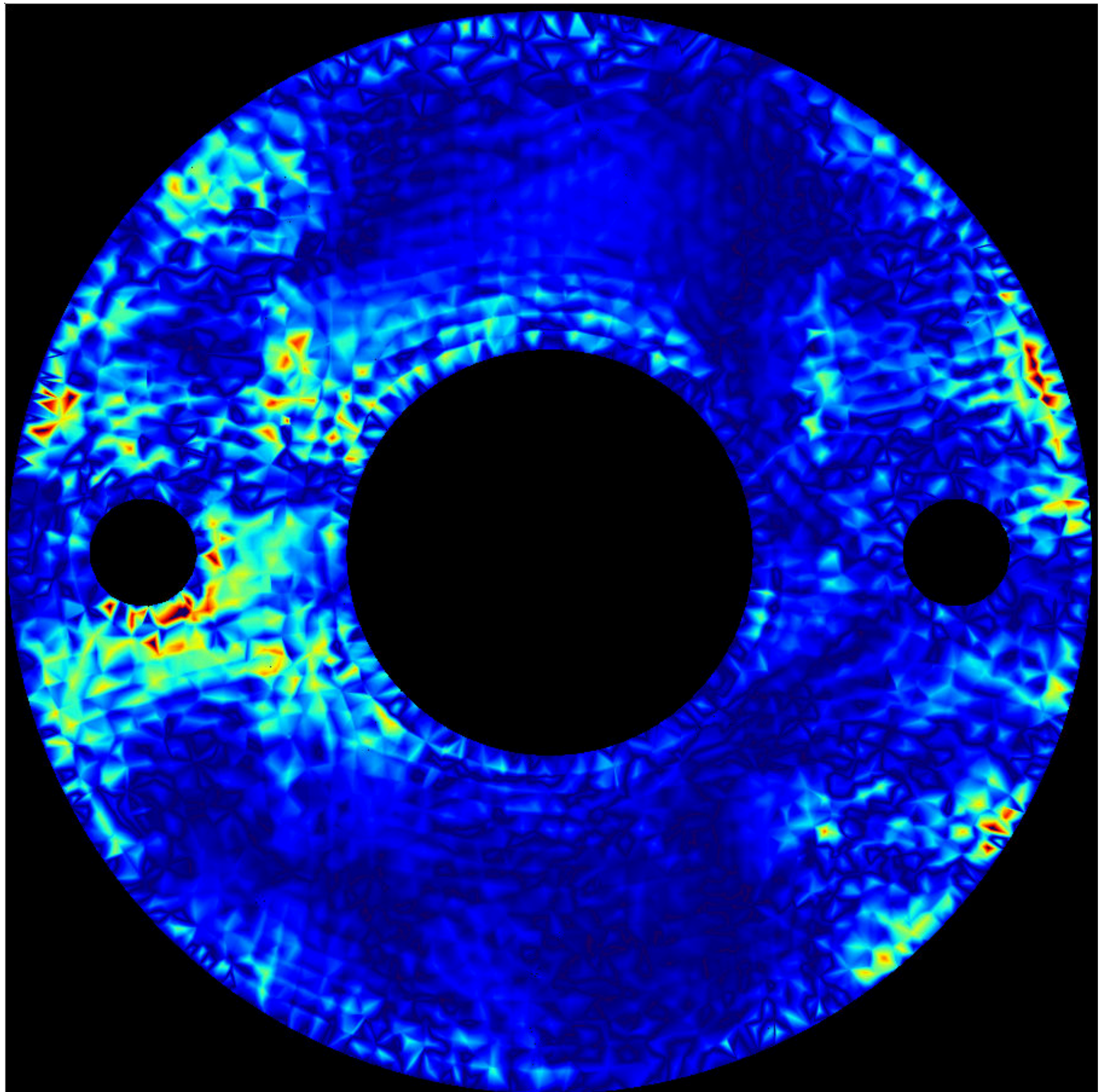
Visualization of slice difference

Visualization is free (textures are available on the GPU memory)
convenient for debugging or investigating FEM behavior



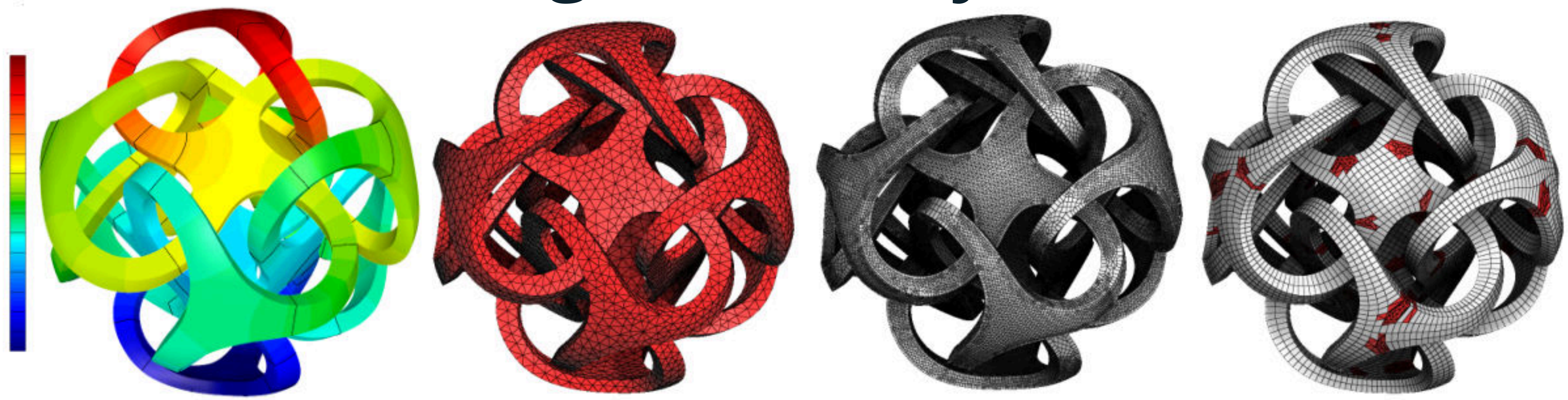
Q1 difference P1

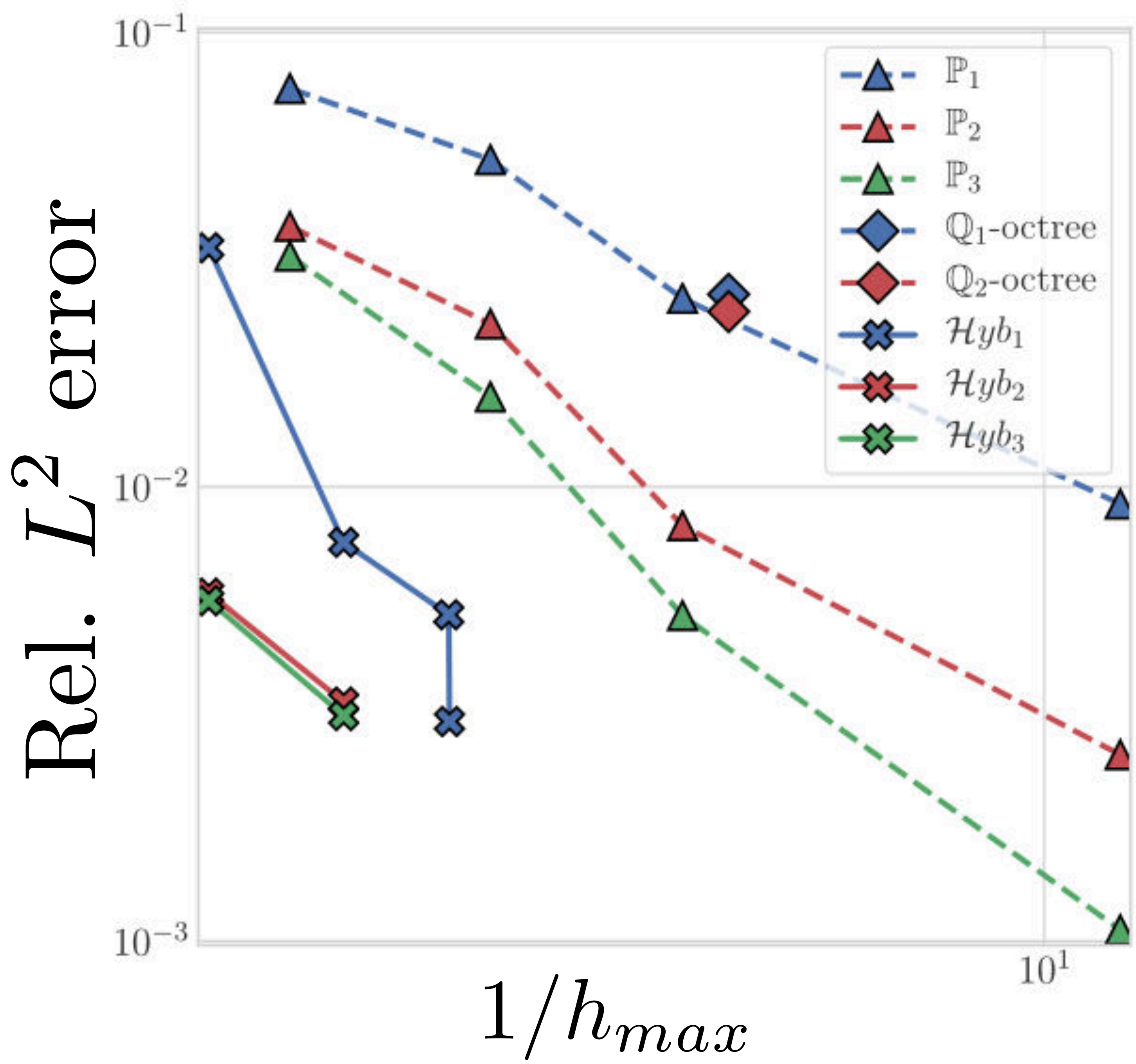
Visualization of slice difference

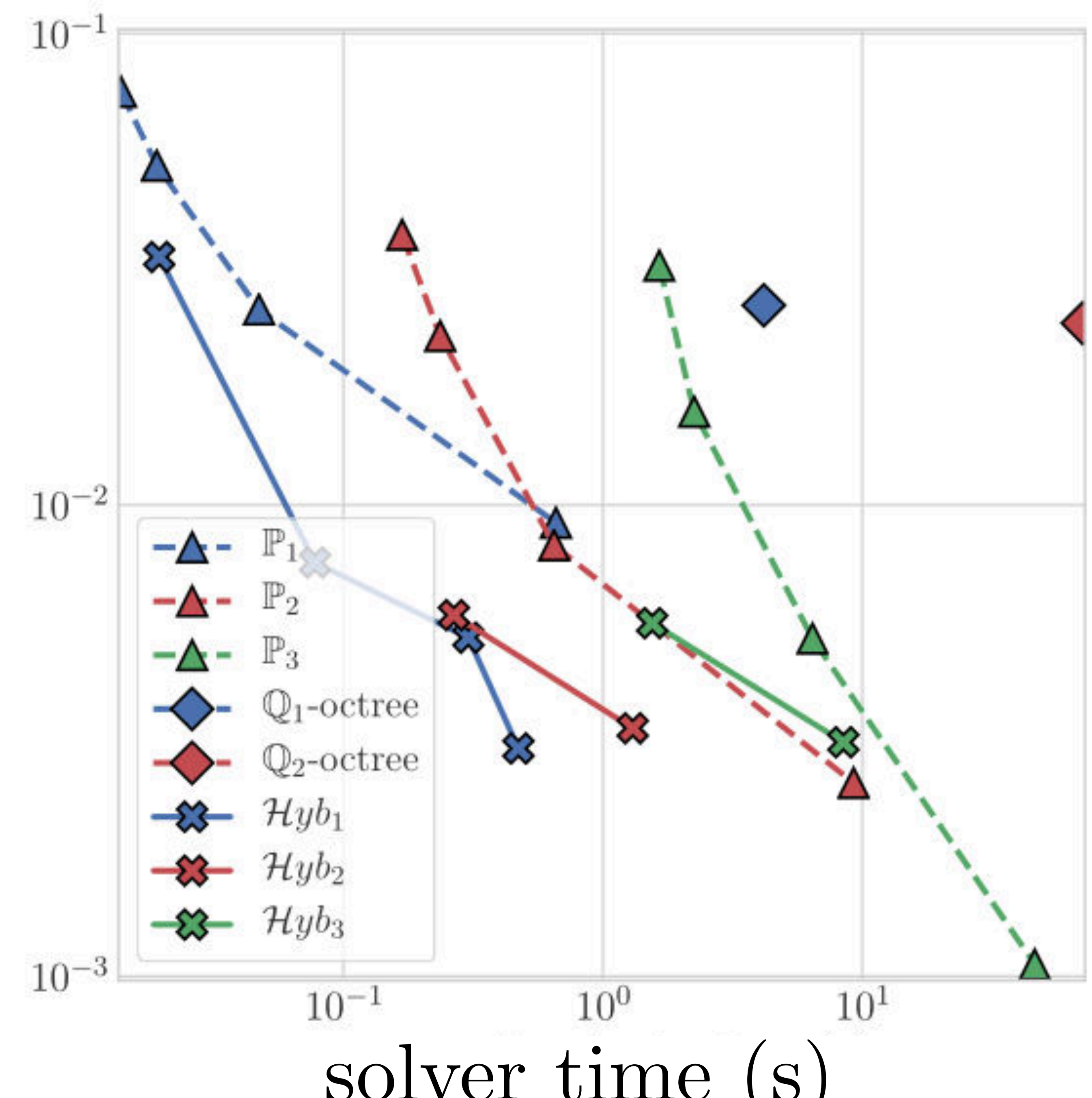


Difference:
Hex-Tet P2/Q1 vs P1

Application: convergence analysis on 3D models







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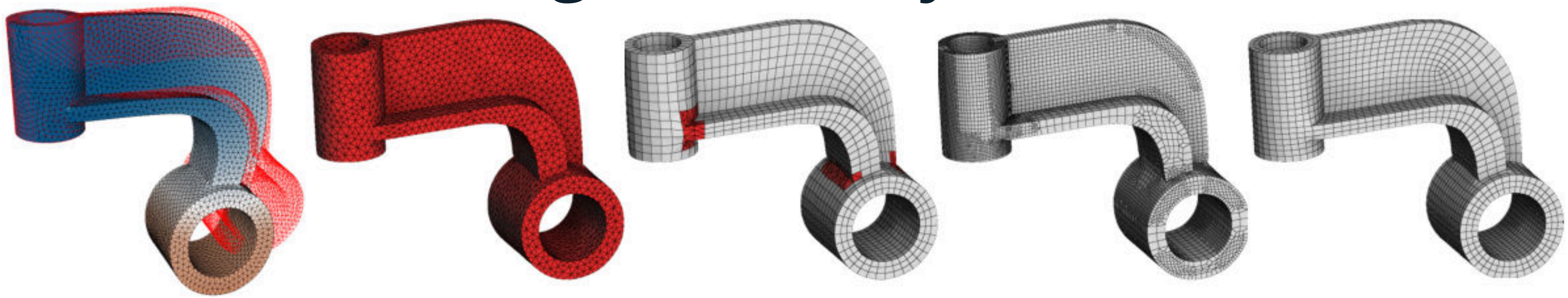

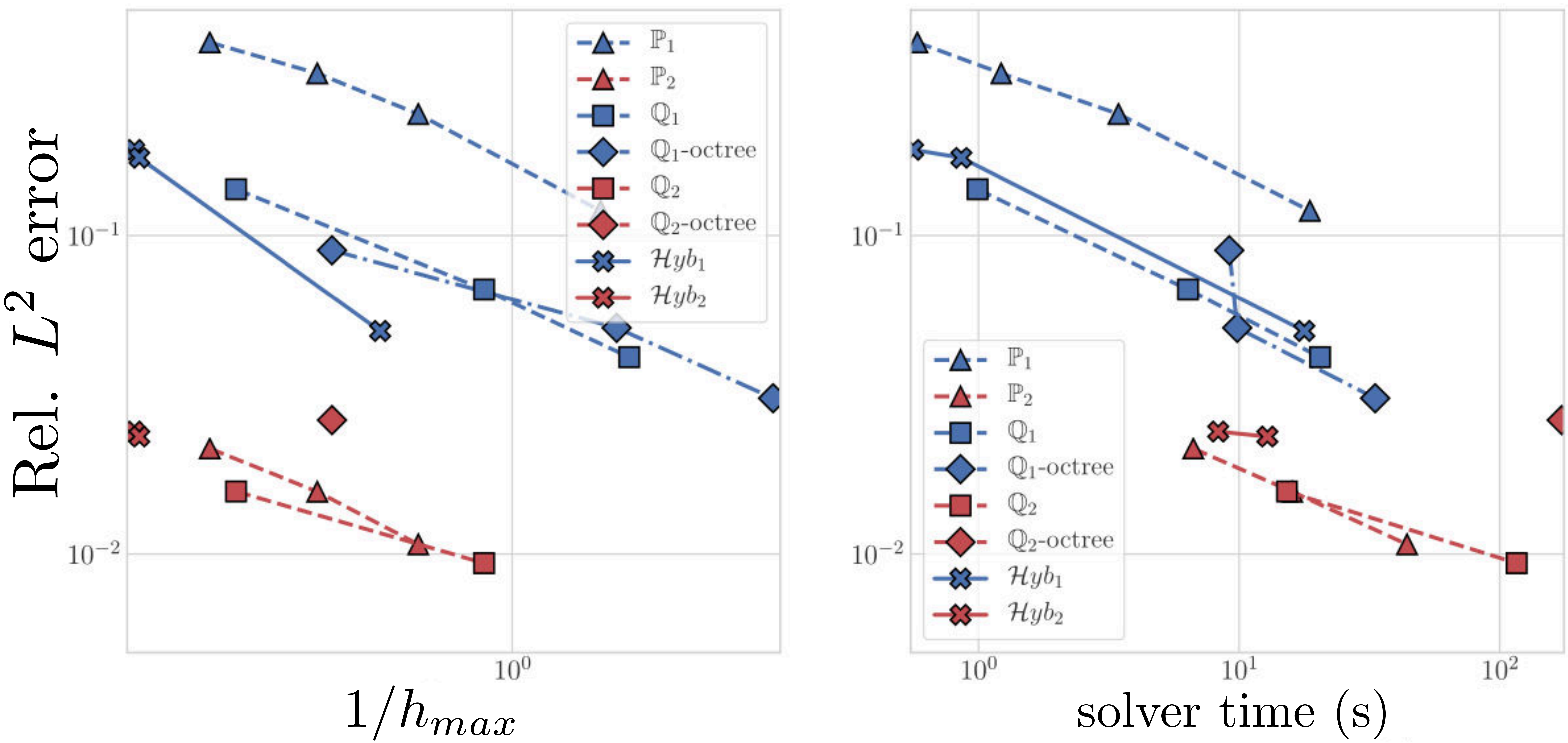

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More information

- Article (detailed explanation):

Reberol, M., and Lévy, B., Computing the distance between two finite element solutions defined on different 3D meshes on a GPU, *SIAM Journal on Scientific Computing (accepted, 2018)*

- Software: github.com/mxncr/FFES, file format:

```
{
    "version": "0.1",
    "groups": [
        {
            "mesh_dim": <int: dimension of the mesh, should be 3>,
            "primitive": <string: TET or HEX, other decompositions not impleme
            "nb_mesh_cp_per_cell": <int: nb of control points per cell in the
            "mapping": <string: mapping function>,
            "mesh_ctrl_points": <string: Base64 encoding of mesh coefficients>
            "field_dim": <int: dimension of the field, e.g. 1 for scalar field
            "nb_field_cp_per_cell": <int: nb of control points per cell in the
            "interpolation": <string: interpolation function>,
            "field_ctrl_points": <string: Base64 encoding of field coefficient
        },
        {
            // another group definition if multiple ElementGroup
        }
    ],
    "arbitrary_key": <string: other metadata can be stored in this json, they
}
```

Mapping and interpolation included in file format

```
vec3 element_mapping(const vec3 ref_pos, const vec3 values[4]){
    vec3 result =
        (1.0f-ref_pos.x-ref_pos.y-ref_pos.z) * values[0]
        + ref_pos.x                          * values[1]
        + ref_pos.y                          * values[2]
        + ref_pos.z                          * values[3];
    return result;
}
```

Support for arbitrary mapping/interpolation: write the GLSL functions in the input file

- Limitations: OpenGL 4.2., memory, 32bits float rasterization

Conclusion

- A fast (real-time) and flexible approach to compute distances between FEM solutions

- Usage: increase parameters (samples, subdivision) to verify convergence of the distance computation

Perspectives

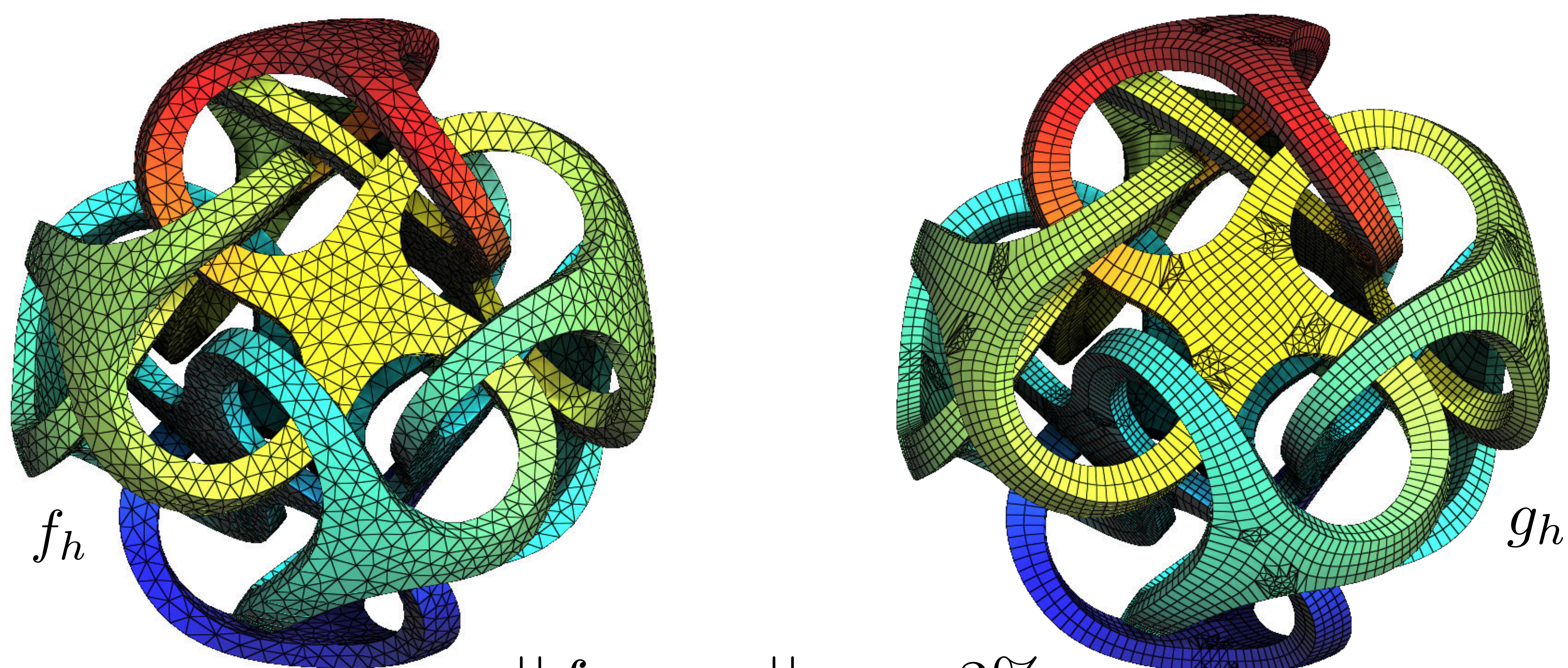
- Integration into Graphite/Geogram for better interactive visualization

- More efficient curved cell mappings (Newton-like correction)

- CPU implementation for portability

Thank you for your attention

Questions ?



$$||f_h - g_h||_{L^2} = 3\%$$

References

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Code Verification by the Method of Manufactured Solutions, 2002

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Rendering planar cuts through quadratic and cubic finite elements, 2004

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GPU-Based Interactive Cut-Surface Extraction From High-Order Finite Element Fields

[AK91] Akio, Doi and Koide, Akio

An efficient method of triangulating equi-valued surfaces by using tetrahedral cells, 1991